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A 3-D MULTIGRID FINITE ELEMENT METHOD FOR THE EULER EQUATIONS

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FOR THE EULER EQUATIONS

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Abstract : The problem of extending the multigrid principle to unstructured tetrahedrizations is solved by a cell-grouping method combined to an upwind finite volume formulation ; an explicit Runge-Kutta solver is applied at each level.

Application to transonic 3-D Euler flows are presented.

UNE METHODE MULTIGRILLE EN ELEMENTS FINIS

POUR LA RESOLUTION DES EQUATIONS D'EULER

EN TROIS DIMENSIONS

Résumé : L'extension du principe "multigrille" à des tétraédrisation non structurées est réalisée en groupant des volumes finis et en appliquant un schéma spatial décentré. La méthode de résolution utilisée à chaque niveau est du type Runge-Kutta. On présente quelques applications à des écoulements transsoniques tridimensionnels.

PAPIER RECUPERÉ ET RECYCLÉ

INTRODUCTION

The solution of the Euler equations on unstructured grids has been recently studied by several teams [LOH1, ANG1, DON, HUG, JAM1].

The motivation in deriving such methods is the perspective of a somewhat more reliable use for the industrial applications. Now one important condition for this use is the possibility to perform heavy 3-D calculations, involving the several dozen thousand nodes that are necessary for the study of complex recirculating flow.

To reach this purpose, it seems mandatory to study solution algorithms that satisfy the following properties

- A cost that increases slowly with the number of nodes
- A low storage requirement
- The ability to vector and parallel computing

These properties are more or less satisfied by explicit multigrid algorithms,that have been studied in the unstructured context by Lohner-Morgan [LOH2] and Angrand-Leyland [ANG2] with Lax-Wendroff-types basic schemes, and Mavriplis-Jameson [MAV], Perez et al. [PER] starting from Jameson's Finite Volume Central-Differenced scheme ; upwind adaptations were also derived in [LAL1]. The main feature by which these studies are different is the way to consider the coarse levels when the grid is unstructured ; we distinguish four ways :

1- the coarse levels are generated separately [LOH2,MAV],

2- fine levels are derived by global or adaptive (local) division from a coarse initial grid [BAN,ANG2],

3- coarse levels are constructed algebrically from a matrix [BRA], this has not yet been applied to Euler calculations,

4- coarse levels are generated from non regular finite volume grouping relying on neighboring relations [LAL2].

We refer to [LAL2] for a discussion of the advantages and disadvantages of each way.

The presented "topological" approach is of the fourth type, and presents the following features :

- the grids are embedded ;

- the coarse meshes are not classical FEM triangulations but generalized finite volume partitions ;

- the spatial approximation is derived on each level ;

- a full approximation storage (non linear) scheme is employed.

The approach has been already presented in 2-D in [LAL2]; furthermore, the paper referenced above deal only with 2-D Euler calculations. One main original point of this paper is to consider the important issue that is the practical application to 3-D flow.

1. THE GENERATION OF THE DIFFERENT LEVELS

The objective is to generate coarse levels automatically from an arbitrary unstructured tetrahedrization.

To achieve such degree of reliability, we explore the possibility of grouping together nodes associated with contiguous control volumes. Thus, coarse levels are not produced by a new **tetrahedrization** of the domain: identifying nodes to **control volumes** permits a homogeneous description of the different levels in terms of Finite Volume partitions.

Finite Volume Dual mesh :

Indeed, it has been observed that simplicial (triangles,tetrahedra) Galerkin approximations are equivalent in some sense to adhoc finite volume formulations on specific dual meshes : for the three-dimensional case, the dual mesh is derived using median plans of the elements, that are defined as plans containing (Fig.1) :

- the middle of an edge

- a face centroid

- an element centroid.

Coarsening agglomerating algorithm :

Grouping together control volumes results in a new (coarser) mesh. Repeating the operation allows us to get coarser and coarser levels until sufficiently many levels are obtained.

The coarsening algorithm that we present here is based on neighboring relations (two cells are neighbors if they contain vertices that are neighbors) ; the algorithm reads as follows :

Consider successively every cell.
(1) if the cell C is already included in a group then consider next cell; else: create a new group containing C, and put into this group each cell neighboring C and not already included in a group.
(2) if the new group contains only the cell C, then destroy the group and put cell C in an existent group containing a neighbor of C.
(3) next cell.

Note that this algorithm is formulated independently of the space dimension: the 3-D context brings no modification.

2. GENERALIZED FINITE VOLUME UPWIND SCHEMES

One main feature of the algorithm is that it relies on the construction of a Finite-Volume Method applicable to an arbitrary partition of the computational domain. This construction is presented for a first-order accurate upwind scheme.

2.1. *First-order scheme*

The time-dependent Euler equations are written in conservative form :

$$(1) \quad W_t + F(W)_x + G(W)_y + H(W)_z = 0,$$

in which as usual :

$$W = (\rho, \rho u, \rho v, \rho w, E)$$

where ρ is the density, (u, v, w) the velocity and E the total energy per unit volume.

The upwind finite volume scheme is derived in the simplest manner that one can imagine. We describe it in the context of the usual explicit time-stepping.

Given a cell C_i , the mean value W_i of the dependent variable in this cell is advanced from time level n to time level $n+1$ as follows :

$$(2) \quad area(C_i)[W_i^{n+1} - W_i^n] = -\Delta t \sum_{j \ neighbor \ of \ i} \Phi(W_i^n, W_j^n, \vec{\eta^{ij}})$$

where $\vec{\eta}^{ij}$ is the following metric vector :

$$\eta_x^{ij} = \int_{\partial C_i \cap \partial C_j} \nu_x d\sigma$$

$$\eta_y^{ij} = \int_{\partial C_i \cap \partial C_j} \nu_y d\sigma$$

$$\eta_z^{ij} = \int_{\partial C_i \cap \partial C_j} \nu_z d\sigma$$

$\vec{\nu} = (\nu_x, \nu_y, \nu_z)$, unit normal vector pointing outward from C_i

and where Φ is a flux-splitting consistent with $\eta_x^{ij} F + \eta_y^{ij} G + \eta_z^{ij} H$.

In this work, we use the van Leer flux splitting [LEE], that is written

$$(3) \quad \Phi_{vl}(W_L, W_R) = F_{vl}^+(W_L) + F_{vl}^-(W_R)$$

in which, when $\vec{\eta} = (1,0,0)$, the F_{vl}^+ and F_{vl}^- are defined as follows

$$(4) \quad F_{vl}^+(W) = \begin{cases} F & si\ M \geq 1 \\ \begin{pmatrix} +\frac{\rho c}{4}(\frac{u}{c}+1)^2 = f_1^+ \\ \frac{f_1^+}{\gamma}((\gamma-1)u+2c) \\ f_1^+ v \\ f_1^+ w \\ f_1^+ \left[\frac{((\gamma-1)u+2c)^2}{2(\gamma^2-1)} + \frac{v^2+w^2}{2}\right] \end{pmatrix} & si\ |M| < 1 \\ 0 & si\ M \leq 0 \end{cases}$$

and

$$(5) \quad F_{vl}^-(W) = F(W) - F_{vl}^+(W)$$

where c is the local sound speed and M the local Mach number : $M = \frac{u}{c}$; for an arbitrary $\vec{\eta}$, the above splitting is applied in a rotated coordinate system [FEZ1].

On the finest mesh, that is a standard triangulation, a second order scheme can be applied: this scheme, a MUSCL-FEM one [FEZ2], is obtained by replacing in the above flux summation the first-order accurate integration

$$(6) \quad \Phi(W_i^n, W_j^n, \vec{\eta^{ij}})$$

by a second-order accurate one

$$(7) \quad \Phi(W_{ij}^n, W_{ji}^n, \vec{\eta^{ij}})$$

that is obtained as follows: a linear interpolation is performed

$$(8) \quad \begin{cases} W_{ij}^n &= W_i^n + \frac{1}{2} \vec{\nabla}_{lim} W(i) \cdot \vec{ij} \\ W_{ji}^n &= W_j^n + \frac{1}{2} \vec{\nabla}_{lim} W(i) \cdot \vec{ji} \end{cases}$$

in which

$$\vec{\nabla}_{lim} W(i) = (D_{lim}^x W(i), D_{lim}^y W(i))$$

is a limited estimate for the nodal gradient defined as:

$$D_{lim}^x W(i) = MinMod(\frac{\partial W}{\partial x})$$

where the MinMod is taken over the tetrahedra that have node i as a vertex and the notation $\frac{\partial W}{\partial x}$ holds for the element-wise P_1-Galerkin derivative.

2.2. *Stability*

The efficiency, and to some extent the robustness of the algorithm relies on the accurate estimation of the maximal time-step ; this is particularly essential when **local time-stepping** is employed.

Unfortunately, estimating the local time-step evaluated from a simplified Fourier analysis can be very hazardous. Hence, we prefer to evaluate a lower bound based on the L^∞ stability of a multi-dimensional model. We refer to [LAL2] for a stability study of linear multidimensional advection model that is also valid in the 3-D case. We define the following "reference time-step" which will be computed on each cell as follow :

$$\Delta t_i = volume(C_i) / (\lambda_{\max}^i \int_{\partial C_i} d\sigma)$$

with

$$\lambda_{\max}^i = \max(\lambda_i \ , \ \max_{j \ neighbor \ of \ i} \lambda_j)$$

and

$$\lambda_i = (u_i^2 + v_i^2 + w_i^2)^{\frac{1}{2}} + c_i$$

where u_i, v_i, w_i, c_i hold for the values in cell C_i of the velocity components and sound speed.

Because of the ratio area/volume in 3-D, a factor of six may be lost w.r.t. usual L^2 stability condition; in practice, time-step larger than Δt_i by a factor of 5 can be used (L^2-)stably and a good strategy for multi-gridding is to set Δt in the range of $4\Delta t_i$ to $5\Delta t_i$.

3. MULTIGRID SCHEME

3.1. *Basic Iteration Method*

A Runge-Kutta scheme is applied with either one or (following [JAM2]), four time-steps :

$$(9) \quad \begin{cases} W^{(0)} = W^n \\ W^{(1)} = W^{(0)} - \alpha_1 \Delta t \;\; D(W^{(0)}) \\ W^{(2)} = W^{(1)} - \alpha_2 \Delta t \;\; D(W^{(1)}) \\ W^{(3)} = W^{(2)} - \alpha_3 \Delta t \;\; D(W^{(2)}) \\ W^{(n+1)} = W^{(3)} - \alpha_4 \Delta t \;\; D(W^{(3)}) \end{cases}$$

where D holds for the spatial fluxes divided by the cell area, see (2).

In the second option, the following coefficients are employed (see [TUR], [LAL3] :

$$(10) \quad \begin{cases} \alpha_1 = .11 \\ \alpha_2 = .2766 \\ \alpha_3 = .5 \\ \alpha_4 = 1. \end{cases}$$

3.2. *FMG first-order scheme*

The basic algorithm uses FAS iterations and Full Multigridding as in [JAM2]. We have it sketched for 3 grids. The transfer operators are defined in the present approach as follows :

- Fine-to-coarse : values are averaged in a conservative manner.

- Coarse-to-fine : the trivial injection is applied.

3.3. *Second-order version*

The second-order spatial scheme is introduced into the fine-grid solver only for the last phase of the full-multigrid process. This introduces a minor modification in the algorithm. However, two disadvantages appear in this construction : first the coarse level correction is less consistent with the fine level smoother ; second, in a full-multigrid approach, the third phase starts from a first-order (medium level) solution instead of a second-order one. The resulting convergence degradation is about a factor 2 and is studied in [LAL4].

4. NUMERICAL ILLUSTRATION

We have experimented the above algorithms with three calculations that require many time steps before reaching steadiness when an explicit scheme is applied.

4.1. *Comparison with a classical approach*

In order to validate the approach,we consider the first-order accurate version. The test case is the classical flow in a channel with 4.2 % thick bump with a Mach at infinity equal to .85.

We recall a first result that was performed by M-H. Lallemand [LAL1] with (what we call) **a two- dimensional multi-triangulation algorithm** : three triangulations are nested standardly (by element division) and as spatial scheme, the first-second-order upwind scheme is applied at each level and the one-step RK1 time stepping is applied at each level and the inter-grid transfers are Galerkin standard ones. For the 2-D calculation, the successive nested triangulations contain respectively 161, 585, and 2225 vertices.

Secondly the presented 3-D algorithm is applied to a tetrahedrization which is essentially a 72x21x3 mesh, see Fig. 2-a. The automatic coarsening gives three coarse levels with the following sizes: 3016 cells, 516 cells, 91 cells; the size ratio between two successive levels tends to be about 6 because of the quasi- 2-D character of the fine mesh; in this severe condition, the CPU ratio between one MG cycle and one usual RK4 iteration is only 1.7.We present in Fig. 2-b the history of the convergence when a Full four-grid algorithm is applied ; it favourably compares to the 2-D 3-grid multi-triangulation calculation (Fig.2-c); this proves that both the original coarsening algorithm and intergrid transfers presented in this paper do not

bring any trouble in 3-D at least when spatially first-order accurate schemes are used.

4.2. *Two-jet flow in chamber*

A practical illustration of the efficiency of the 3-D code is the calculation of two impinging supersonic jets in a rocket combustion chamber. This test case is rather severe because of high pressure ratios, resulting in Mach number as high as 5. A mesh with 21,859 vertices is used and the first-order scheme with RK4 iteration is applied. The convergence in Full Multi-Grid mode is rather fast with the first-order accurate MG scheme: 36 cycles are sufficient for a residual division by nearly 1000, the reduction factor is .89, see Fig. 3. Unfortunately, attempts to apply the second-order accurate MG solver made arise instabilities (negative pressures), while the fine grid iteration stayed stable : we must remember that the multigrid schemes are designed to be L^2 stable, but not L^∞ stable (in the sense of positiveness or maximum principle).

4.3. *Flow past a delta wing*

We turn now to a second-order accurate calculation; then the full-MG strategy cannot be applied since only first-order accurate solutions are available on coarse levels: this does not give a very good initial condition; another strategy consist in the combination of the MG algorithm with the use of a succession of locally refined meshes. We illustrate this with the following calculation: the flow around a delta wing with circular profile is considered; the Mach number at infinity is 1.2 and the angle of attack is equal to 10 degrees; mesh adaption by local element division is applied after a coarse mesh (about 5000 vertices) calculation; the final fine mesh contains 22160 vertices; the numerical solution is obtained by applying the second-order MG version (fine level) relying on the RK4 time-stepping (at each level).

We give in Fig.4 an idea of the solution and the history of convergence with multigrid when we use as initial condition an interpolation of the solution obtained with the coarse mesh. The convergence is much less good than in Sec.4.2; the reduction factor has a mean of .89 over 60 cycles, but it degrades progressively. However, in this calculation, a very partial convergence seems to be sufficient since the solution does not essentially change after the first 40 cycles.

4.4. *Computational costs*

The calculations have been performed on a mini-computer in scalar mode. For giving an idea of the complexity (number of operations) of the algorithm, we compare the different measured costs with the cost of a standard one-grid RK4

solver, that involves the calculation of a local time-step and four Euler fluxes defined in Sec.2:

-Cost of the coarsening algorithm: about 0.2 RK4.

-Cost of one MG cycle: about 1.7 RK4.

The second number is slightly lower in the case of the second-order scheme since second-order is applied only to the fine grid. Both numbers do validate the coarsening approach for both start up and transfers.

5. CONCLUSION

Using the presented algorithm, an efficient calculation can be performed on an arbitrary 3-D tetrahedrization without any need of taking care of the coarse levels: they are generated in an automated manner once for all and we can just ignore them.

While the problem of coarsening seems to us rather well solved by this approach, the choice of the iterative method is not completely satisfactory: the second-order version is not yet efficient enough and presents some lack of robustness. One way to deal with the difficulty in the second-order extension can rely on the Defect Correction principle; this possibility is currently studied by B. Koren and M.H. Lallemand [KOR]. We may also consider some more sophisticated iterative methods such as implicit ones already investigated in 2-D [LAL4].

6. ACKNOWLEDGEMENTS

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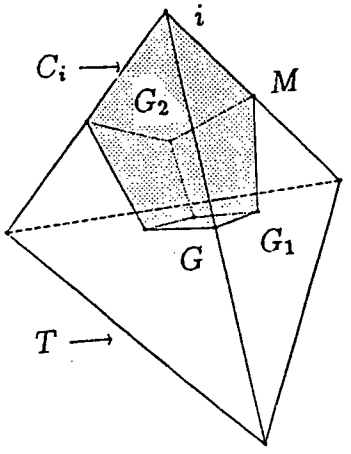
(a)



i : node

T : tetrahedron

C_i : cell around i

M : middle of edge

G : face centroid

G_1, G_2 : face centroids

(b)

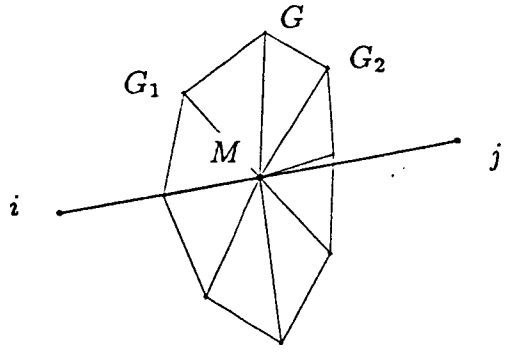
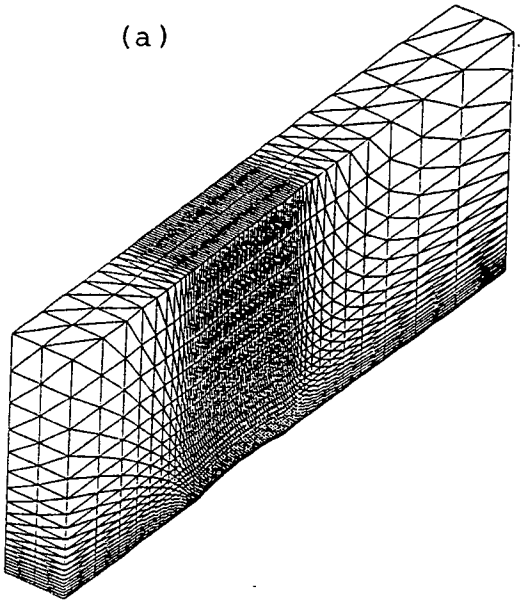
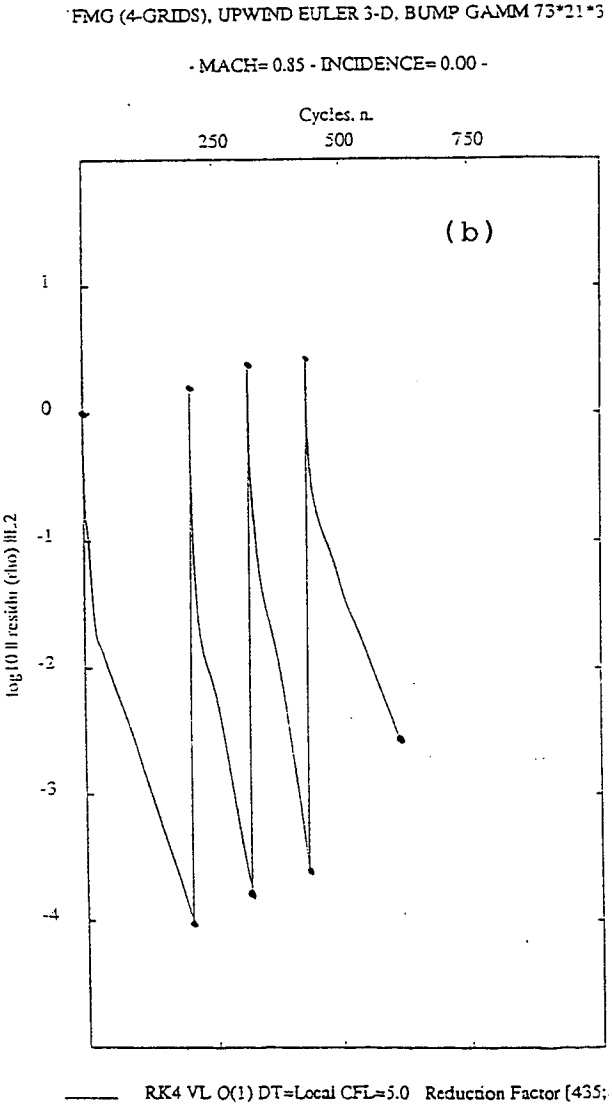
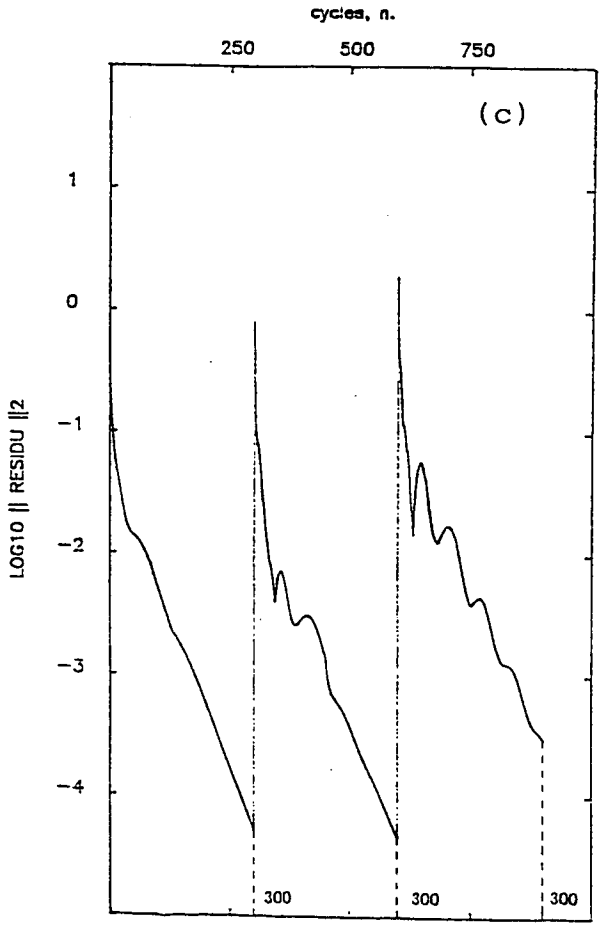


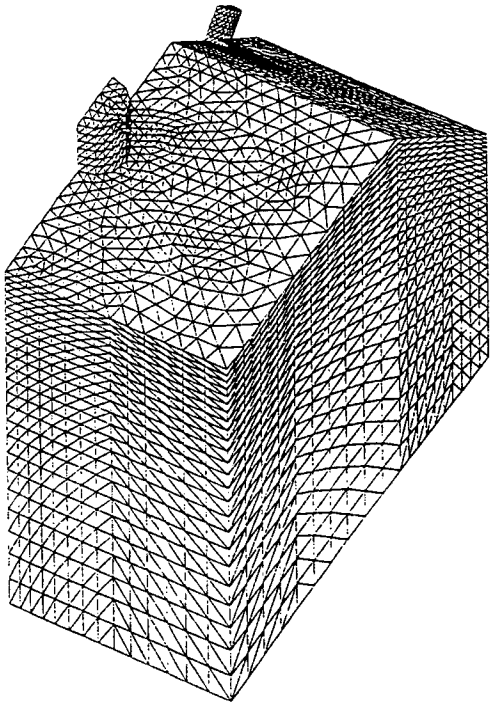
Figure 1 : Sketchy representation of the Finite-Volume partition :
1a : intersection of C_i and an element T
1b : intersection of C_i and C_j.

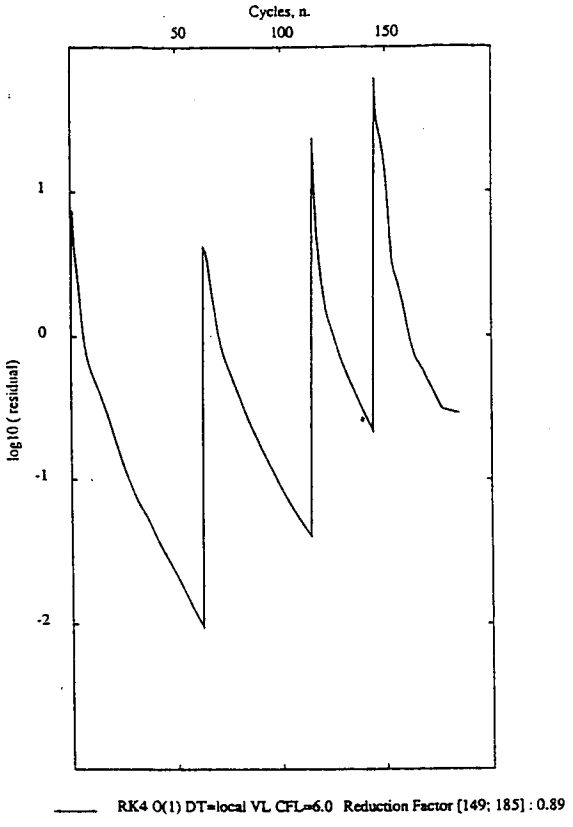


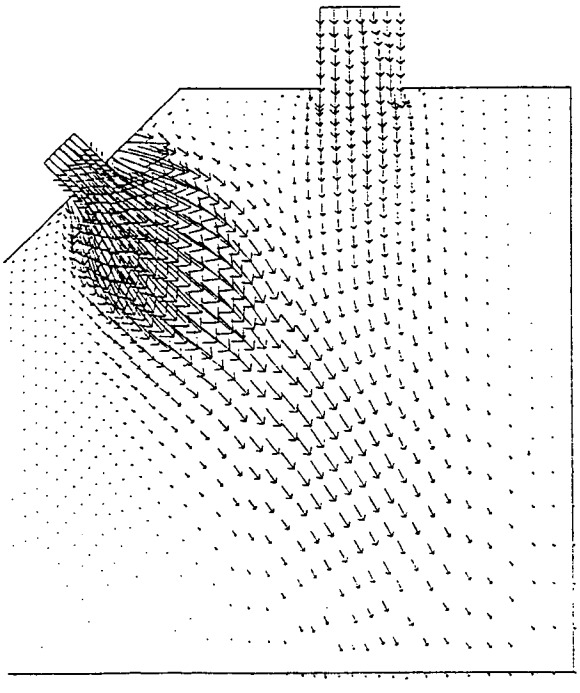
Figures 2a to 2c :
First-order 3-D scheme ;
a comparison of 3-D FMG (a,b)
with a 2-D multitriangulation
analog (c) (LAL1).







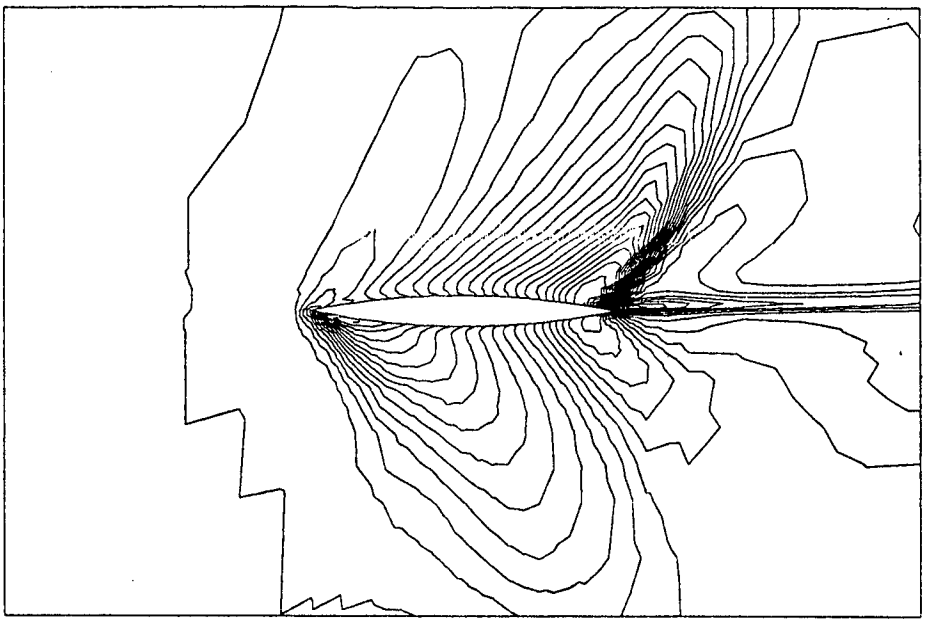




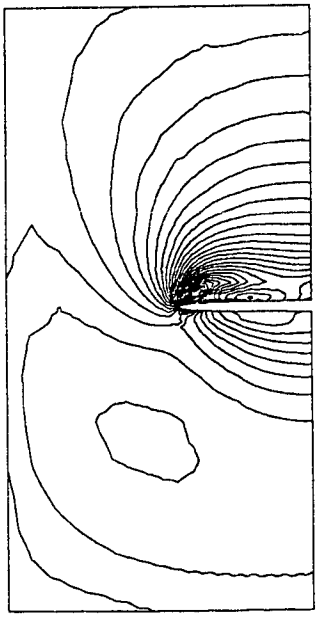
Figures 3a to 3c :
First-order 3-D scheme :
Transonic impinging jets in a combustion chamber (21 000 nodes)

ITERATION= 170

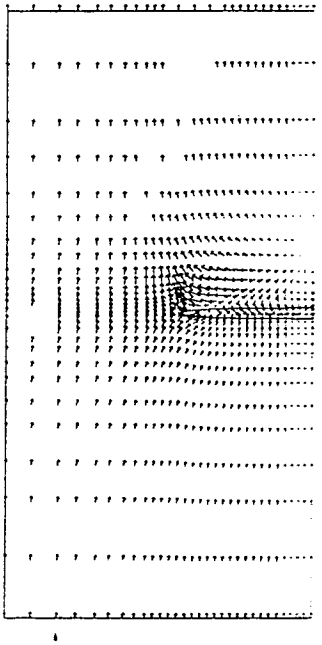
(c)



(d)

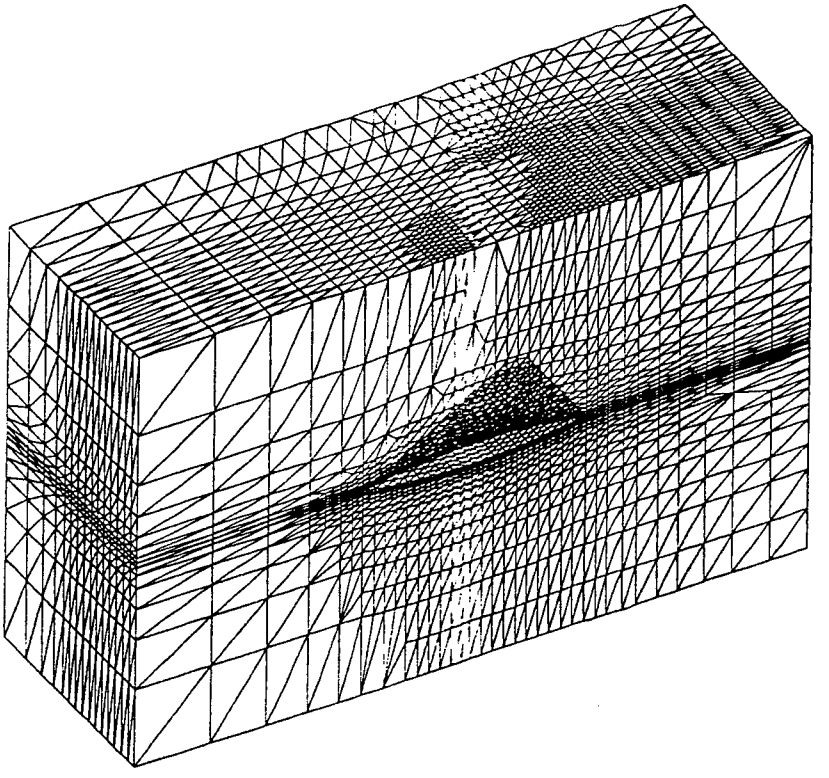


(e)

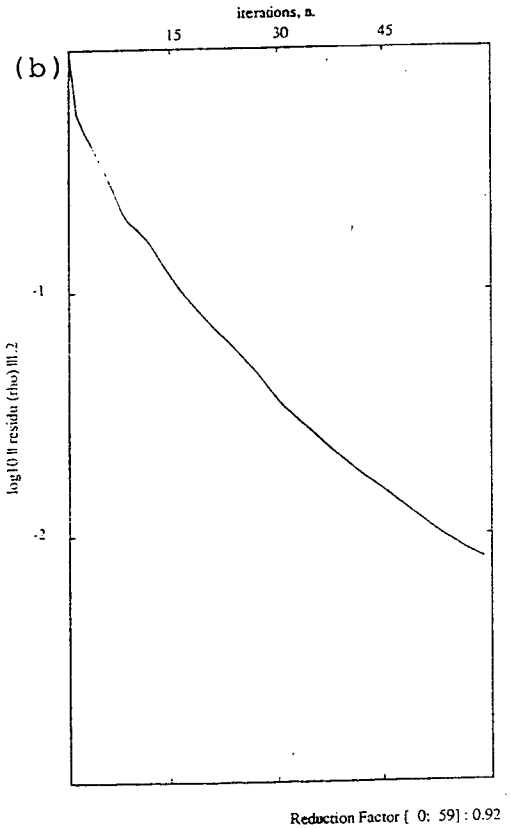


Figures 4c, 4d, 4e
Flow past a delta wing :
4c : Mach contours on the symmetry plane
4d : Mach contours on a normal plane at 90% of the chord
4e : The corresponding velocity field.

(a)



(b)



Figures 4a and 4b
Flow past a delta wing :
4a : sketch of the enriched mesh (20000 nodes)
4b : convergence history, when initial conditon is the coarse mesh solution.

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